TOPOLOGY QUALIFYING EXAM– SPRING 2023

Instructions: Solve any FOUR out of the problems. If you work on all problems, your top four problem scores will count towards your exam grade.

You have two hours to work on the exam. **NOTE:** In your solutions you should not use theorems from the book that are outside the scope of the material covered in this course. **Show your work; all answers must be justified appropriately.**

Problem 1.

(1) Prove that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ for $m \neq n$.
(2) Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ (for $m \neq n$) be open sets. Show that U is not homeomorphic to V.

Problem 2. State and prove the Brouwer fixed point theorem.

Problem 3. Let F_n be the free group on n letters.

(1) Show that if X is an index 3 subgroup of F_2 then X is isomorphic to F_4. (Hint: Euler characteristic may be helpful)
(2) Find a subgroup $G \subset F_2$, with $G \cong F_5$.
(3) If $n \neq m \geq 1$, show F_n is not isomorphic to F_m.

Problem 4.

(1) Show that any continuous map $S^2 \to S^1 \times S^1$ is nullhomotopic.
(2) Compute the homology groups $H_n(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.
(3) Compute the groups $H_n(X, A)$ where X is a closed orientable surface of genus two and A is the (middle) circle shown.

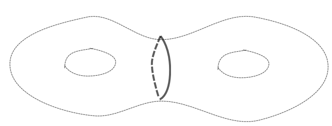

FIGURE 1. The curve A

Problem 5.

(1) Let $T = S^1 \times S^1$ be the torus. Describe a subspace $X \cong S^1 \vee S^1$ of $T = S^1 \times S^1$ so that the induced map $H_1(X) \to H_1(T)$ is an isomorphism. Show that $H_1(X) \cong \mathbb{Z}^2$.
(2) Suppose that the subspace X you chose in (1) has a neighborhood U inside of T so that U deformation retracts to X. Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. Hint: Use the long exact sequence of a pair.

(3) True or False (justify your answer): For every matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

there exists a homeomorphism $\varphi\colon T \to T$ so that the induced map $\varphi_*\colon H_1(T) \to H_1(T)$ is A (with respect to the basis you chose in (1)).